

# Construction of SU(3) irreps in canonical SO(3)-coupled bases

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Alternative canonical methods for defining canonical SO(3)-coupled bases for SU(3) irreps are considered and compared. It is shown that a basis that diagonalizes a particular linear combination of SO(3) invariants in the SU(3) universal enveloping algebra gives basis states that have good  $K$  quantum numbers in the asymptotic rotor-model limit.

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## I. INTRODUCTION

A common problem in the construction of group or Lie algebra representations is to define a canonical basis in situations where multiplicities occur. For example, bases which reduce the subgroup chain

$$\begin{array}{ccccc} \text{SU}(3) & \supset & \text{SO}(3) & \supset & \text{SO}(2) \\ (\lambda\mu) & K & L & & M \end{array} \quad (1)$$

are indexed by the quantum numbers  $(\lambda\mu)$ ,  $L$  and  $M$ , of the respective groups SU(3), SO(3), and SO(2). However, an extra label  $K$  is required to distinguish different irreps of SO(3) that occur in a given SU(3) irrep. This paper, is concerned with useful ways to define orthogonal sets of such SO(3) irreps.

In principle, multiple occurrence of subgroup irreps can be defined in any arbitrary way. However, it is useful to have a well-defined “canonical” definition, that can be reproduced by anyone so that results derived by one person are meaningful to someone else. For example, in applications of group representations, considerable use is made of Clebsch-Gordan coupling and Racah recoupling coefficients that are defined for particular resolutions of the multiplicities that occur. Two kinds of multiplicity arise: one is the multiplicity in the choice of basis for each irrep. Another, of equal importance, is the multiplicity of different irreps that occur in the decomposition of tensor products of irreps. In this paper, we address the resolution of the first of these two multiplicities.

## II. SU(3) IRREPS AND THEIR ASYMPTOTIC LIMITS

There are several ways to construct SU(3) irreps in an SO(3)-coupled basis and derive the corresponding matrices representing elements of the su(3) Lie algebra. Note that we use upper case letters to denote a Lie group, e.g., SU(3), and lower case letters, e.g., su(3), for its Lie algebra. The su(3) Lie algebra is spanned by five components of a quadrupole tensor  $\mathcal{Q}$  and three components of an angular momentum  $\mathbf{L}$ . As we discuss below, su(3) has an asymptotic limit in which it contracts to the Lie algebra, rot(3), of a rigid-rotor model. The latter Lie algebra is likewise spanned by five components of a quadrupole tensor  $\mathcal{Q}$  and three components of an angular momentum  $\mathbf{L}$ . Both su(3) and rot(3) have commutation relations

$$[L_k, L_{k'}] = -\sqrt{2} (1k, 1k' | 1k + k') L_{k+k'}, \quad (2)$$

$$[L_k, \mathcal{Q}_\nu] = -\sqrt{6} (1k, 2\nu | 2\nu + k) \mathcal{Q}_{\nu+k}. \quad (3)$$

However, they differ in the commutators of their  $\{\mathcal{Q}_\nu\}$  operators;

$$[\mathcal{Q}_\nu, \mathcal{Q}_\mu] = 3\sqrt{10} (2\mu, 2\nu | 1\mu + \nu) L_{\mu+\nu} \times \begin{cases} 0 & \text{for rot(3)} \\ 1 & \text{for su(3)} \end{cases} \quad (4)$$

Thus, whereas su(3) is semi-simple, its contraction, rot(3), is a semi-direct sum of an abelian subalgebra, isomorphic to  $\mathbb{R}^5$ , and an so(3) angular momentum algebra; we denote this by writing  $\text{rot(3)} \simeq [\mathbb{R}^5]\text{so(3)}$ .

As shown in ref. [1], basis states for a generic  $(\lambda, \mu)$ , irrep of su(3), are labeled by angular-momentum quantum numbers,  $L$  and  $M$ , with  $L$  running over the values

$$L = \begin{cases} \lambda + K, \lambda + K - 1, \dots, K & \text{for } K \neq 0 \\ \lambda, \lambda - 2, \dots, 0 \text{ or } 1 & \text{for } K = 0 \end{cases} \quad (5)$$

with

$$K = \mu, \mu - 2, \dots, 0 \text{ or } 1. \quad (6)$$

Thus, in the generic case, there is a multiplicity of states with given values of  $L$  and  $M$ , which can be indexed by  $K$  or any other convenient label.

Irreps of the type  $(\lambda, 0)$  are particularly simple. They have orthonormal  $\text{SO}(3)$ -coupled bases given, without multiplicity, by a set of states

$$\{|LM\rangle; M = -L, \dots, +L, L = \lambda, \lambda - 2, \dots, 0 \text{ or } 1\}, \quad (7)$$

in which  $L$  runs over even or odd integer values according as  $\lambda$  is, respectively, even or odd. Reduced matrix elements for such multiplicity-free irreps have analytical expressions given [2, 3] (in natural units) by the equations

$$\langle L \| \mathcal{Q} \| L \rangle = \sqrt{2L+1} (L0, 20 | L0) (2\lambda + 3), \quad (8)$$

$$\langle L+2 \| \mathcal{Q} \| L \rangle = \sqrt{2L+1} (L0, 20 | L+2, 0) [4(\lambda - L)(\lambda + L + 3)]^{\frac{1}{2}}. \quad (9)$$

A systematic way to derive matrix elements for a generic  $\text{SU}(3)$  irrep was given [2, 3] in terms of vector coherent state [4, 5] theory. VCS methods were also used in a derivation of  $\text{SU}(3)$  Clebsch-Gordan coefficients in an  $\text{SO}(3)$ -coupled basis [6, 7]. Conversely, a set of  $\text{SU}(3)$  Clebsch-Gordan coefficients computed in an  $\text{SO}(3)$ -coupled basis enables one to derive the  $\text{SO}(3)$ -reduced matrices of the  $\text{SU}(3)$  quadrupole tensor in that basis. Examples of reduced matrix elements derived in this way are given below. Such methods do not give analytical expressions for generic irreps, for which there are multiplicities. However, analytical expressions are obtained [2, 3] in the asymptotic limits which are approached as either  $\lambda$  or  $\mu \rightarrow \infty$ .

In the following, we restrict consideration to  $\text{su}(3)$  irreps  $\{(\lambda, \mu)\}$  with  $\lambda \geq \mu$ . This is because of the well-known fact (shown, for example, in [3]) that the irreps  $(\lambda, \mu)$  and  $(\mu, \lambda)$  are simply related. Specifically, if  $\Gamma_\nu^{(\lambda\mu)}$  denotes the matrix representing the quadrupole operator  $\mathcal{Q}_\nu$  in the  $\text{su}(3)$  irrep  $(\lambda, \mu)$ , then

$$\Gamma_\nu^{(\lambda\mu)} = -\Gamma_\nu^{(\mu\lambda)}. \quad (10)$$

With this restriction, asymptotic expressions for the  $\text{su}(3)$  quadrupole matrix elements are given for  $\lambda \rightarrow \infty$  by

$$\langle KL \| \mathcal{Q} \| KL \rangle \sim \sqrt{2L+1} [(LK, 20 | LK)(\Lambda + \delta_{K1}\sigma_{LL})] \quad (11)$$

$$\langle KL+1 \| \mathcal{Q} \| KL \rangle \sim \sqrt{2L+1} [(LK, 20 | L+1K) \sqrt{[(\Lambda - L - 1 + \delta_{K1}\sigma_{L+1,L})(\Lambda + L + 1 + \delta_{K1}\sigma_{L+1,L})]}] \quad (12)$$

$$\langle KL+2 \| \mathcal{Q} \| KL \rangle \sim \sqrt{2L+1} [(LK, 20 | L+2K) \sqrt{[(\Lambda - 2L - 3 + \delta_{K1}\sigma_{L+2,L})(\Lambda + 2L + 3 + \delta_{K1}\sigma_{L+2,L})]}] \quad (13)$$

$$\begin{aligned} \langle K+2, L' \| \mathcal{Q} \| KL \rangle &= (-1)^{L'-L} \langle KL \| \mathcal{Q} \| K+2, L' \rangle \\ &\sim \sqrt{(2L+1)(1 + \delta_{K,0})} (LK, 2, \pm 2 | L', K \pm 2) \sqrt{\frac{3}{2}(\mu - K)(\mu + K + 2)}, \end{aligned} \quad (14)$$

where  $\Lambda = 2\lambda + \mu + 3$  and

$$\sigma_{L'L} = \frac{1}{2}(\mu + 1)(-1)^{\lambda+L} \times \begin{cases} -\frac{3L(L+1)}{3-L(L+1)} & \text{for } L' = L \\ L+1 & \text{for } L' = L+1 \\ -L & \text{for } L' = L-1 \\ -1 & \text{for } L' = L \pm 2 \end{cases} \quad (15)$$

These asymptotic expressions are shown below to provide accurate approximate expressions for  $\text{su}(3)$  matrix elements for moderately large but finite values of  $\lambda$ . They are similar in form to those of an irrep of the  $\text{rot}(3)$  rigid-rotor algebra), given by

$$\begin{aligned} \langle KL' \| \mathcal{Q} \| KL \rangle &= \sqrt{2L+1} [(LK, 20 | L'K) \bar{q}_0 \\ &\quad + \delta_{K,1} (-1)^{\lambda+L+1} (L, -1, 22 | L'1) \bar{q}_2], \end{aligned} \quad (16)$$

$$\begin{aligned} \langle K+2, L' \| \mathcal{Q} \| KL \rangle &= (-1)^{L'-L} \langle KL \| \mathcal{Q} \| K+2, L' \rangle \\ &= \sqrt{(2L+1)(1 + \delta_{K,0})} (LK, 22 | L', K+2) \bar{q}_2, \end{aligned} \quad (17)$$

with

$$\bar{q}_0 = 2\lambda + \mu + 3, \quad \bar{q}_2 = \sqrt{\frac{3}{2}} \mu. \quad (18)$$

The latter expressions give accurate approximations when both  $\lambda$  and  $\mu$  are large but are generally not as accurate as those of eqns. (11) - (14).

A computationally simple method [8], used in the present calculations, for deriving numerically precise matrix elements of an  $\text{su}(3)$  irrep is to start from two known irreps,  $(\lambda_1, 0)$  and  $(\lambda_2, 0)$ , and diagonalize the  $\text{SO}(3)$ -invariant operator  $\mathcal{Q} \cdot \mathcal{Q}$  in the tensor product of these irreps, where  $\mathcal{Q} := \mathcal{Q}^{(1)} + \mathcal{Q}^{(2)}$  is the summed quadrupole tensor for the two irreps. To within a term proportional to the  $\text{SO}(3)$  Casimir invariant,  $\mathbf{L} \cdot \mathbf{L}$ , the operator  $\mathcal{Q} \cdot \mathcal{Q}$  is proportional to the  $\text{SU}(3)$  Casimir invariant. Thus, its eigenstates belong to  $\text{SU}(3)$  irreps and, in the process of deriving them, one obtains all the reduced matrix elements of the quadrupole tensor (albeit in a basis chosen arbitrarily by the computer). However, as shown in ref. [8], if one then diagonalizes the operator  $\mathcal{Q}^{(1)} \cdot \mathcal{Q}^{(2)}$  within the space of a  $(\lambda, \mu)$  irrep within the tensor product of  $(\lambda_1, 0)$  and  $(\lambda_2, 0)$  irreps, then the degeneracies are lifted and the multiplicity of  $\text{SO}(3)$  irreps is resolved. Simple techniques for constructing such basis states and deriving their matrix elements were given in ref. [8] and are used in the present calculations. Examples of reduced quadrupole matrix elements obtained in this way for the (32,5) and (10,4) irreps are shown in the columns of Tables I and II labeled  $\mathcal{Q}^{(1)} \cdot \mathcal{Q}^{(2)}$ . However, this bases does not appear to correspond to any of the canonical bases we consider below.

### III. ALTERNATIVES FOR RESOLVING THE $\text{SO}(3)$ MULTIPLICITIES

We consider three alternatives.

#### A. Alternative I

A standard way to resolve the  $\text{SU}(3) \supset \text{SO}(3)$  multiplicity is by eigenstates of the angular-momentum-zero coupled operator

$$X_3 := (\mathbf{L} \otimes \mathcal{Q} \otimes \mathbf{L})_0. \quad (19)$$

This operator is an  $\text{SO}(3)$  scalar in the  $\text{SU}(3)$  universal enveloping algebra [9]. Its potential use for resolving the  $\text{SU}(3) \supset \text{SO}(3)$  multiplicity was noted by Bargmann and Moshinsky [10]. Such a use is easily implemented because matrix elements of  $X_3$  in any  $\text{SO}(3)$ -coupled basis for an  $\text{SU}(3)$  irrep are given to within an unimportant  $L$ -dependent constant,  $c_L$ , by

$$\langle \beta L' \| X_3 \| \alpha L \rangle = \delta_{L', L} c_L \langle \beta L \| \mathcal{Q} \| \alpha L \rangle. \quad (20)$$

Thus, an  $\text{SO}(3)$ -coupled basis that diagonalizes  $X_3$  is given by the eigenstates of the  $M^L$  matrices with elements

$$M_{\beta\alpha}^L := \langle \beta L \| \mathcal{Q} \| \alpha L \rangle. \quad (21)$$

A variant of this method was used in the construction of bases for VCS irreps by  $K$ -matrix methods [11, 12]. Examples of reduced quadrupole matrix elements in such a basis are given in Tables I and II.

#### B. Alternative II

A second alternative is to use generally accepted  $\text{SU}(3)$  Clebsch-Gordan coefficients in an  $\text{SO}(3)$ -coupled basis to derive reduced matrix elements of the  $\text{SU}(3)$  quadrupole operator by means of the identity

$$\langle \beta L' \| \mathcal{Q} \| \alpha L \rangle = \left[ \frac{4}{3} (2L' + 1) (\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu) \right]^{\frac{1}{2}} ((\lambda\mu)\alpha L; (11)2 \| (\lambda\mu)\beta L'), \quad (22)$$

where  $(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu)$  is proportional to the value of the  $\text{SU}(3)$  Casimir operator for the  $(\lambda, \mu)$  irrep and  $((\lambda\mu)\alpha L; (11)2 \| (\lambda\mu)\beta L')$  is an  $\text{SO}(3)$ -reduced  $\text{SU}(3)$  Clebsch-Gordan coefficient.

In principle, the resolution of the  $\text{SU}(3) \supset \text{SO}(3)$  multiplicity, defined in this way, is only canonical to the extent that the Clebsch-Gordan coefficients are themselves expressed relative to a canonical basis. But, even if they are

TABLE I: Comparison of quadrupole reduced matrix elements  $\langle K_f L_f || \mathcal{Q} || K_i L_i \rangle$  for bases defined by diagonalizing the operator  $\mathcal{Q}^{(1)} \cdot \mathcal{Q}^{(2)}$  and by the I, II and III alternatives, as defined in the text, for the SU(3) irrep (32, 5). Values given by the asymptotic approximations of eqns. 11-15 are shown in the column headed A.S. Values for rot(3), given by eqns. 16-17, are shown in the column headed ROT(3). Values obtained by the alternative algebraic approach described in Ref. [8], corresponding to the quadrupole-quadrupole interaction strength  $C=3.99$ , are in the last column.

$K_i L_i$	$K_f L_f$	$\mathcal{Q}^{(1)} \cdot \mathcal{Q}^{(2)}$	I	II	III	A.S.	ROT(3)
1; 3	1; 1	58.030319	81.421678	81.979149	81.974076	81.975606	81.610661
3; 3	1; 1	59.131052	15.313707	11.975740	12.010416	12	10.606602
1; 3	1; 2	-71.091833	-81.321756	-80.940834	-80.945466	-80.944425	-79.5
3; 3	1; 2	-40.223759	7.666261	10.980970	10.946736	10.954451	9.682458
1; 3	1; 3	-1.248765	-61.784810	-61.476192	-61.482530	-61.481705	-63.531095
3; 3	1; 3	-86.819713	0	7.551035	7.473012	7.483315	6.614378
3; 3	3; 3	62.730470	123.266515	122.957882	122.964235	122.963409	122.9634092
1; 4	1; 2	90.971036	99.475476	100.490358	100.480593	100.484540	101.737583
3; 4	1; 2	44.061450	17.991295	10.900948	10.990508	10.954451	9.682458
1; 4	1; 3	-66.087883	-52.793962	-51.839707	-51.849622	-51.845926	-51.693575
3; 4	1; 3	69.315355	14.450569	12.953794	12.961550	12.961481	11.456439
1; 4	3; 3	-0.916687	3.009507	-3.8839466	-3.792147	-3.794733	-3.354102
3; 4	3; 3	-99.882581	-127.061082	-127.590584	-127.588524	-127.589968	-127.787323
1; 4	1; 4	-97.415026	-108.055071	-107.316055	-107.334482	-107.331699	-105.038286
3; 4	1; 4	-38.133331	0	10.407601	10.277606	10.297396	9.101698
3; 4	3; 4	28.612663	39.252708	38.513682	38.532119	38.529328	38.529328
1; 5	1; 3	85.179826	126.274052	128.974225	128.958781	129.336770	129.037785
1; 5	3; 3	-0.269230	-3.674329	1.317411	1.274925	1.264911	1.118034
3; 5	1; 3	60.505815	26.583826	10.495438	10.647742	10.583005	9.354143
3; 5	3; 3	65.665141	69.189233	69.022209	69.009663	69.229088	69.558608
5; 5	3; 3	-76.363826	16.305010	14.425467	14.515278	14.491377	16.201852
1; 5	1; 4	-83.593639	-69.460806	-67.269069	-67.283818	-67.278526	-65.453419
1; 5	3; 4	82.77925	9.338480	-5.939770	-5.807912	-5.796551	-5.123475 1
3; 5	1; 4	-16.878049	13.723799	14.185688	14.212501	14.198591	12.549900
3; 5	3; 4	-68.306769	-135.235541	-136.270512	-136.278511	-136.280593	-136.610395
5; 5	3; 4	68.027894	5.913498	9.628153	9.450810	9.486833	10.606602
1; 5	1; 5	-48.716838	-95.531881	-93.682063	-93.716833	-93.712654	-96.231811
3; 5	3; 5	43.126580	-10.404218	-12.110821	-12.089411	-12.091955	-12.091955
5; 5	5; 5	81.164960	181.510801	181.367616	181.380946	181.379330	181.379330
3; 5	1; 5	-52.087438	0	12.414743	12.299948	12.313845	10.884004
5; 5	3; 5	113.765385	0	5.258647	5.006770	5.038315	5.633007

not, provided they are freely available, they serve the practical purpose of making it possible to compare the results of calculations by different researchers who use a common set of such coefficients. For present purposes, we use the Clebsch-Gordan coefficients of refs. [6, 7]. Some results are shown for comparison with the other alternatives in Tables I and II. The comparisons show a remarkable similarity between these results and those of the following alternative. This will be explained in the Discussion.

### C. Alternative III

A third alternative is given by basis states which diagonalize a specified linear combination of the SO(3) scalar operators  $X_3$  and

$$X_4 := (\mathbf{L} \otimes [\mathcal{Q} \otimes \mathcal{Q}]_2 \otimes \mathbf{L})_0 \quad (23)$$

within the space of an SU(3) irrep.

The rationale for choosing a particular linear combination is based on the observation that there is a natural resolution of the  $\text{SO}(3) \subset \text{SU}(3)$  multiplicity in the contraction limit in which an irrep of the  $\text{su}(3)$  algebra progresses asymptotically towards an irrep of the rotor model algebra, denoted  $\text{rot}(3)$ . In particular, as pointed out in ref. [13], the intrinsic quadrupole moments of a  $\text{rot}(3)$  irrep, for which there is a naturally-defined SO(3)-coupled basis, are related to the SO(3) invariants  $\bar{X}_3$  and  $\bar{X}_4$ , where the latter operators are defined, as for the corresponding  $\text{su}(3)$  operators

TABLE II: Comparisons of quadrupole reduced matrix elements  $\langle K_f L_f || Q || K_i L_i \rangle$  as described in Table I for the SU(3) irrep (10, 4).

$K_i L_i$	$K_f L_f$	$\mathcal{Q}^{(1)} \cdot \mathcal{Q}^{(2)}$	I	II	III	A.S.	ROT(3)
0; 2	0; 0	19.146198	25.227104	26.854801	26.823096	26.832816	27
2; 2	0; 0	20.625775	12.473680	8.415442	8.515925	8.485281	6.928203
0; 2	0; 2	-25.280167	-33.827282	-32.203990	-32.280594	-32.271172	-32.271172
2; 2	0; 2	-22.476614	0	10.353197	10.111788	10.141851	8.280787
2; 3	0; 2	32.612214	19.722620	13.305980	13.464860	13.416407	10.954451
2; 3	2; 2	-30.272798	-39.887555	-42.461171	-42.411040	-42.426407	-42.690748
0; 4	2; 3	20.939094	10.108085	-9.182851	-8.552679	-8.485281	-6.928203
2; 4	2; 3	-32.438270	-41.477548	-41.183272	-41.375935	-41.366653	-41.828220
4; 4	2; 3	-18.966068	5.276214	8.367394	8.079759	8.197561	8.197561
0; 4	0; 2	29.639536	34.848900	41.886357	41.742972	41.815923	43.296321
0; 4	2; 2	0.058923	-3.976255	2.594292	2.343090	2.267787	1.851640
2; 4	0; 2	19.653177	22.023088	8.545059	9.080523	8.783101	7.171372
2; 4	2; 2	21.468704	28.265009	26.965058	26.958674	26.992062	27.947655
4; 4	2; 2	30.379909	12.821131	10.867348	11.055995	10.954451	10.954451
0; 4	0; 4	-45.391345	-48.446517	-40.877618	-41.340626	-41.281422	-41.281423
2; 4	2; 4	11.845141	-9.673158	-16.863115	-16.470046	-16.512569	-16.512569
4; 4	4; 4	33.546210	62.755119	57.740733	57.810677	57.793992	57.793992
2; 4	0; 4	-11.594159	0	15.396745	15.025873	15.073844	12.307742
4; 4	2; 4	-33.333631	0	5.247793	4.721849	4.854239	4.854239

$X_3$  and  $X_4$ , but in terms of the commuting rot(3) quadrupole operators. Because of the understood contraction of  $\text{su}(3) \rightarrow \text{rot}(3)$  for large values of  $\lambda$ , these observations suggested similar relationships for  $\text{su}(3)$ . Further relationships between the rigid-rotor model and the SU(3) model were developed by Leschber and Draayer [14].

The  $\text{su}(3) \rightarrow \text{rot}(3)$  contraction is derived as follows [15]. Let

$$\epsilon(\lambda\mu) := \frac{1}{2} \left[ \lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu \right]^{-\frac{1}{2}}, \quad (24)$$

denote the inverse square root of the eigenvalue of the SU(3) Casimir invariant

$$C_2 := \mathcal{Q} \cdot \mathcal{Q} + 3\mathbf{L} \cdot \mathbf{L} \quad (25)$$

for the irrep  $(\lambda, \mu)$ , and let  $Q$  denote the  $\text{su}(3)$  quadrupole tensor in inverse units of  $\epsilon(\lambda\mu)$ , i.e. the tensor with components

$$Q_\nu := \epsilon(\lambda\mu) \mathcal{Q}_\nu. \quad (26)$$

It follows from eqn. (22) that, for values of  $L \ll \lambda$ , the non-zero reduced matrix elements of  $Q$  are of order of magnitude

$$\langle \beta L' || Q || \alpha L \rangle \lesssim \left[ \frac{4}{3} (2L' + 1) \right]^{\frac{1}{2}}. \quad (27)$$

Moreover, the rhs of the commutation relation

$$[Q_\nu, Q_\mu] = 3\sqrt{10} (2\mu, 2\nu | 1\mu + \nu) \epsilon(\lambda\mu)^2 L_{\mu+\nu}, \quad (28)$$

becomes negligible when used with states of angular momentum  $L$  for which

$$\epsilon(\lambda\mu)^2 L \ll 1. \quad (29)$$

Thus, within the subspace of states of angular momentum  $L$  for which eqn. (29) is satisfied, the matrix elements of an  $\text{su}(3)$  irrep become indistinguishable from those of a  $\text{rot}(3)$  irrep. In this situation,  $\text{su}(3)$  is said to contract to  $\text{rot}(3)$ .

This contraction is of considerable interest in nuclear physics for explaining the origins of rotational structure in terms of the nuclear shell model in an  $\text{SU}(3) \supset \text{SO}(3)$  coupled basis. Because there is a natural resolution of the  $\text{SO}(3) \subset \text{ROT}(3)$  multiplicity, the  $\text{su}(3) \rightarrow \text{rot}(3)$  contraction, is also significant for the resolution of the  $\text{SO}(3) \subset \text{SU}(3)$  multiplicity.

The results of ref. [13] suggest that the above defined basis states of the  $\text{rot}(3)$  algebra should diagonalize a linear combination of the  $\bar{X}_3$  and  $\bar{X}_4$  operators. To ascertain that this is true and determine the linear combination, we consider the ratios of the matrix elements

$$R(L, K) := \frac{\langle K+2, L \| \bar{X}_4 \| KL \rangle}{\langle K+2, L \| \bar{X}_3 \| KL \rangle} = \frac{\langle K+2, L \| [\mathcal{Q} \otimes \mathcal{Q}]_2 \| KL \rangle}{\langle K+2, L \| \mathcal{Q} \| KL \rangle} \quad (30)$$

for  $\text{rot}(3)$  irreps. Equations (16) and (17) give the reduced  $\text{rot}(3)$  matrix elements  $\langle KL \| \mathcal{Q} \| KL \rangle$  and  $\langle K+2, L \| \mathcal{Q} \| KL \rangle$ . From them, we can evaluate

$$\langle K+2, L \| [\mathcal{Q} \otimes \mathcal{Q}]_2 \| KL \rangle = \sum_{K_1 L_1} U(L_2 L_2; L_1 2) \frac{\langle K+2, L \| \mathcal{Q} \| K_1 L_1 \rangle \langle K_1 L_1 \| \mathcal{Q} \| KL \rangle}{\sqrt{2L_1+1}} \quad (31)$$

and the ratio  $R(KL)$  for any values of  $L$  and  $K$ . In this way, it is determined that  $R(LK)$  takes the  $L$ - and  $K$ -independent value

$$R(LK) = \sqrt{\frac{8}{7}} \bar{q}_0. \quad (32)$$

This result means that the basis states of the rigid-rotor  $\text{rot}(3)$  algebra with good  $K$  quantum numbers are eigenstates of the  $\text{SO}(3)$ -invariant

$$\bar{Z} := \bar{X}_4 - \sqrt{\frac{8}{7}} \bar{q}_0 \bar{X}_3. \quad (33)$$

Similarly, we can define basis states for an  $\text{SU}(3)$  irrep to be eigenstates of the corresponding  $\text{SO}(3)$ -invariant

$$Z := X_4 - \sqrt{\frac{8}{7}} (2\lambda + \mu + 3) X_3, \quad (34)$$

with the expectation that, in such a basis, the  $\text{su}(3)$  quadrupole matrix elements between states of  $L \ll \lambda$  will approach those of a  $\text{rot}(3)$  irrep in the asymptotic limit. Such basis states are uniquely defined, and provide a physically relevant resolution of the  $\text{SO}(3)$  multiplicity for any  $\text{SU}(3)$  irrep. Results obtained for such  $\text{SU}(3)$  bases are shown in Tables I and II.

#### IV. DISCUSSION

Tables I and II show comparisons of reduced quadrupole matrix elements obtained for the alternatives given above for defining orthonormal  $\text{SO}(3)$ -coupled basis states for  $\text{SU}(3)$  irreps. The tables also show the corresponding results given by the asymptotic approximation of eqns. (11) - (15) and for the  $\text{rot}(3)$  matrix elements given by eqn. (16) and (17). It should be emphasized that the results given for the  $\text{SU}(3)$  matrix elements listed in the columns headed I, II, and III are all numerically accurate to the precision shown; they only differ to the extent that they were computed relative to different bases. The asymptotic results in the column headed A.S. are expected to agree with those of column III for values of  $\lambda \gg L_i$ . Those listed in the column headed  $\text{ROT}(3)$  are for the  $\text{rot}(3)$  rotor algebra and likewise are expected to approximate those of columns III and A.S. when both  $\lambda \gg L_i$  and  $\mu \gg L_i$ .

It can be seen that alternatives II and III are the same to within 1-3% and, as expected, both are consistent with an approach to the results of the asymptotic limit for large values of  $\lambda$ . The results of diagonalizing the  $\text{SO}(3)$  invariant,  $X_3$ , in alternative I are qualitative similar but it is clear that the eigenstates of the linear combination of  $X_3$  and  $X_4$ , given by  $Z$  in eqn. (34), give results much closer to those of asymptotic rotor-model limit. The equivalence of results II and III is quite remarkable and fortuitous in the sense that it means that the bases used in the calculation of  $\text{SU}(3) \supset \text{SO}(3)$  Clebsch-Gordan coefficients in refs. [6, 7] can, in fact, be regarded as canonical in the above-defined sense that the basis states are eigenstates of a Hermitian operator. This result was unexpected because the choice of basis states for an  $\text{SU}(3)$  irrep used in the computation of the  $\text{SU}(3)$  Clebsch-Gordan coefficients given in refs. [6, 7], did not make use of the  $\text{SO}(3)$ -invariant operator,  $Z$ . However, the construction of the basis states that were used did make use of rotor-model methods which, in the asymptotic limit likewise give standard  $\text{rot}(3)$  results. Thus, in retrospect, it is understood that the Clebsch-Gordan coefficients obtained should be consistent with the  $\text{SU}(3)$  bases states defined by alternative II.

The tabulated matrix elements also show the expected result that the accurate matrix elements for the basis III are given more accurately, for small values of  $\mu$  by the asymptotic  $SU(3)$  results of column A.S. than by those of the  $ROT(3)$  limit.

In conclusion, we remark that the above results provide a physical and practical resolution of the so-called inner, i.e.,  $SU(3) \supset SO(3)$ , multiplicity problem. However, the outer multiplicity that occurs in the decomposition of tensor products of  $SU(3)$  irreps is also of importance and, at present, we know of no canonical way to resolve it.

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